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Note

Generation of Boundary-Fitted Curvilinear Coordinate Systems for a Two-Dimensional Axisymmetric Flow Problem

I. Introduction

One of the most efficient methods for dealing with fluid mechanical problems having irregularly shaped boundaries is to use a curvilinear coordinate system in which coordinate lines are coincident with all boundaries. Thompson *et al.* proposed a method for automatic numerical generation of such coordinate systems [1], where the coordinates are taken to be the solution of an elliptic equation in the physical plane. In the case of the physical x–y plane, a Laplace equation has been successfully taken as the equation for coordinate transformation, primarily because it exhibits a maximum principle which guarantees that the maximum values of the curvilinear coordinates occur on the boundary of the physical region, and secondly because the equation for the stream function has the same form as a Laplace equation in the physical x–y plane. However, this situation does not hold in the case of cylindrical geometry.

The objective of this note is to apply three possible transformations to the solution of two-dimensional axisymmetric flow problems and to clarify the differences among them.

II. Mathematical Formulation

A. *Coordinate Transformation*

The general transformation from the physical r–z coordinate system to the transformed ξ–η coordinate system is given by $\xi = \xi(r, z)$ and $\eta = \eta(r, z)$, where ξ and η are chosen as solutions of an elliptic equation with Dirichlet boundary conditions. For an axisymmetric geometry, the equations for the transformations may take the following forms including a parameter δ,

$$\frac{\partial^2 \xi}{\partial r^2} + \delta \frac{1}{r} \frac{\partial \xi}{\partial r} + \frac{\partial^2 \xi}{\partial z^2} = P(\xi, \eta) \tag{1a}$$

$$\frac{\partial^2 \eta}{\partial r^2} + \delta \frac{1}{r} \frac{\partial \eta}{\partial r} + \frac{\partial^2 \eta}{\partial z^2} = Q(\xi, \eta), \tag{1b}$$

where P and Q are coordinate control functions. The parameter δ has one of the values, 0, $+1$, or -1. The case with $\delta = 1$ corresponds to a transformation based on a

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solution of a cylindrical Laplace's equation. The case with $\delta = 0$ is an ordinary rectangular–rectangular transformation, in which the effects of cylindrical geometry are not considered explicitly.

Equation (1) in the transformed ξ–η coordinate system is

$$\alpha \frac{\partial^2 r}{\partial \xi^2} - 2\beta \frac{\partial^2 r}{\partial \xi\, \partial \eta} + \gamma \frac{\partial^2 r}{\partial \eta^2} + J^2 P(\xi, \eta) \frac{\partial r}{\partial \xi} + J^2 Q(\xi, \eta) \frac{\partial r}{\partial \eta} = \delta \frac{J^2}{r} \tag{2a}$$

$$\alpha \frac{\partial^2 z}{\partial \xi^2} - 2\beta \frac{\partial^2 z}{\partial \xi\, \partial \eta} + \gamma \frac{\partial^2 z}{\partial \eta^2} + J^2 P(\xi, \eta) \frac{\partial z}{\partial \xi} + J^2 Q(\xi, \eta) \frac{\partial z}{\partial \eta} = 0, \tag{2b}$$

where

$$\alpha = r_\eta^2 + z_\eta^2, \qquad \beta = r_\xi z_\xi + r_\eta z_\eta,$$
$$\gamma = r_\xi^2 + z_\xi^2, \qquad J = r_\xi z_\eta - r_\eta z_\xi.$$

The system described by Eq. (2) is a quasilinear elliptic system for the functions $r(\xi, \eta)$ and $z(\xi, \eta)$ in the transformed plane. Here it should be noted that the term $\delta(J^2/r)$, appears in the right-hand side of Eq. (2a). This term originates from the terms, $\delta(1/r)(\partial\xi/\partial r)$ and $\delta(1/r)(\partial\eta/\partial r)$ in Eq. (1), and can take into account the effect of cylindrical geometry on the coordinate transformation explicitly. The set of Eq. (2) can be solved numerically using second-order central finite difference expressions and a point SOR iteration technique described in [2].

B. *Equations of Axisymmetric Flow*

The stream function–vorticity formulation of the two-dimensional axisymmetric, incompressible viscous flow equations is given by

$$\frac{\partial}{\partial t} \omega + \frac{\partial}{\partial r} (u\omega) + \frac{\partial}{\partial z} (v\omega) = \nu \left\{ \frac{\partial^2 \omega}{\partial r^2} + \frac{\partial^2 \omega}{\partial z^2} + \frac{1}{r} \frac{\partial \omega}{\partial r} - \frac{\omega}{r^2} \right\}, \tag{3a}$$

$$\frac{1}{r} \left(\frac{\partial^2 \Psi}{\partial r^2} + \frac{\partial^2 \Psi}{\partial z^2} - \frac{1}{r} \frac{\partial \Psi}{\partial r} \right) = -\omega, \tag{3b}$$

where Ψ is the stream function, ω the vorticity, ν the kinematic viscosity, u the velocity component in the r direction, and v the velocity component in the z direction.

The transformed equations are given by

$$\frac{\partial \omega}{\partial t} + \frac{1}{J} z_\eta \frac{\partial}{\partial \xi} (u\omega) - \frac{1}{J} z_\xi \frac{\partial}{\partial \eta} (u\omega) - \frac{1}{J} r_\eta \frac{\partial}{\partial \xi} (v\omega) + \frac{1}{J} r_\xi \frac{\partial}{\partial \eta} (v\omega)$$
$$= \frac{\nu}{J^2} \left\{ \alpha \frac{\partial^2 \omega}{\partial \xi^2} - 2\beta \frac{\partial^2 \omega}{\partial \xi\, \partial \eta} + \gamma \frac{\partial^2 \omega}{\partial \eta^2} + J^2 P_1^*(\xi, \eta) \frac{\partial \omega}{\partial \xi} \right.$$
$$\left. + J^2 Q_1^*(\xi, \eta) \frac{\partial \omega}{\partial \eta} - \frac{J^2}{r^2} \omega \right\}, \tag{4a}$$

$$\frac{1}{rJ^2}\left\{\alpha\frac{\partial^2\Psi}{\partial\xi^2}-2\beta\frac{\partial^2\Psi}{\partial\xi\,\partial\eta}+\gamma\frac{\partial^2\Psi}{\partial\eta^2}+J^2P_2^*(\xi,\eta)\frac{\partial\Psi}{\partial\xi}\right.$$
$$\left.+J^2Q_2^*(\xi,\eta)\frac{\partial\Psi}{\partial\eta}\right\}=-\omega, \tag{4b}$$

where the functions P_1^*, Q_1^*, P_2^*, and Q_2^* are defined by

$$P_1^*(\xi,\eta)\equiv P(\xi,\eta)+(1-\delta)\frac{1}{rJ}z_\eta, \tag{5a}$$

$$Q_1^*(\xi,\eta)\equiv Q(\xi,\eta)-(1-\delta)\frac{1}{rJ}z_\xi, \tag{5b}$$

$$P_2^*(\xi,\eta)\equiv P(\xi,\eta)-(1+\delta)\frac{1}{rJ}z_\eta, \tag{5c}$$

$$Q_2^*(\xi,\eta)\equiv Q(\xi,\eta)+(1+\delta)\frac{1}{rJ}z_\xi. \tag{5d}$$

From comparison between Eqs. (1) and (3) when setting the functions $P(\xi,\eta)$ and $Q(\xi,\eta)$ to zero, the following points can be noted with respect to the choice of the parameter δ:

(1) For $\delta=1$, that is, when the coordinate transformation is based on a solution of a Laplace equation in the r–z coordinate system, the functions P_1^* and Q_1^* in the vorticity equation (4a) vanish. Physically, this transformation can be said to be focused on the vorticity.

(2) For $\delta=-1$, the functions P_2^* and Q_2^* in the stream function equation (4b) vanish. When the vorticity is zero, Eq. (4b) has the same form as Eq. (1), and a constant ξ line in the r–z physical plane coincides with one of the streamlines.

(3) For $\delta=0$, the transformation does not depend on whether the physical plane is described by the x–y coordinate systems or by the r–z coordinate system. When the problem to be solved is described using the r–z coordinate system, the effect of the r–z coordinate system is included in the scale factors α, β, γ, and J of the transformed equations.

The above problem for the choice of δ does not arise in the case of a rectangular–rectangular transformation. For axisymmetric flow problems, the choice of $\delta=-1$ may be suitable because of the coincidence between the constant coordinate lines and streamlines. On the other hand, for the thermal conduction problem,

$$\rho C\frac{\partial T}{\partial t}=k\left(\frac{\partial^2T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial T}{\partial Z^2}\right) \tag{6}$$

the choice of $\delta=1$ is preferable because of the coincidence between the constant coordinate lines and constant temperature contours.

The set of Eqs. (4) can be solved using standard finite difference techniques previously described in the literature [3, 4].

III. Results and Discussion

The problem chosen for study is shown in Fig. 1a. Incompressible fluid flows in an annular channel with a trapezoid axisymmetric obstacle. The transformation from the physical to the transformed geometry is indicated schematically in Figs. 1a and b. The irregular outer boundary surface, C–D–E–F–G–H, is transformed to the constant-diameter straight boundary surface, C'–D'–E'–F'–G'–H'.

Three different curvilinear coordinate systems obtained as a solution of Eq. (2) are compared in Fig. 2, where each line corresponds to the constant ξ line or η line in the ξ–η coordinate shown in Fig. 1b. The coordinate control functions, $P(\xi, \eta)$ and $Q(\xi, \eta)$ are set to zero in these transformations. Another point to be noted here regards the boundary condition for solution of Eq. (2). A condition, $\partial r/\partial \eta = 0$, is assigned at the inflow (A–C) and the outflow (B–H) boundaries instead of the Dirichlet conditions.

Comparing the three curvilinear coordinates, it is seen that the constant ξ lines of the $\delta = -1$ transformation are best fitted to the irregular outer boundary, and that the constant ξ lines of the $\delta = 1$ transformation are farthest from the outer boundary. In the case of the $\delta = 0$ transformation, the mesh spacing in the radial direction is constant at the inflow and the outflow boundaries. This result can be automatically obtained using the condition $\partial r/\partial \eta = 0$. In the case of the $\delta = 1$ transformation, the mesh spacing increases with increasing radius. On the otherhand, in the case of the $\delta = -1$ transformation, the mesh spacing decreases with increasing radius. It should

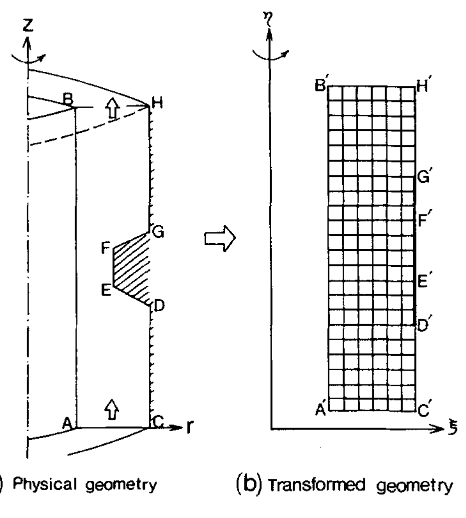

Fig. 1. Test problem—annular flow channel with obstacle.

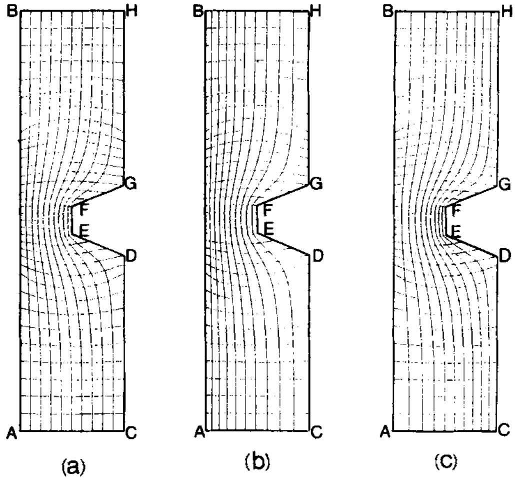

FIG. 2. Comparison of curvilinear coordinate systems. (a) $\delta = 0$ transformation, (b) $\delta = 1$ transformation, and (c) $\delta = -1$ transformation.

be noted that the mesh spacing in the $\delta = -1$ transformation is the same as for equivolume division. This point seems to be desirable in the application of axisymmetric flow problems in order to obtain higher accuracy with a limited number of coordinate lines.

Figure 3 shows the results of velocity vectors calculated using the three different transformations of Fig. 2. The Reynolds number, Re at the inflow is taken as 4×10^4 in the calculations.

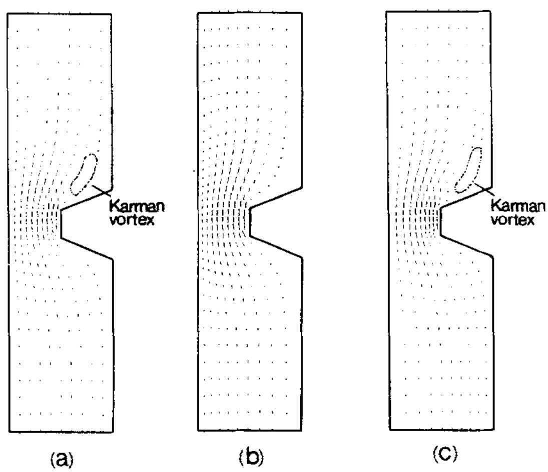

FIG. 3. Velocity vectors calculated using the three different transformations of Fig. 2 ($\mathrm{Re} = 4 \times 10^3$).

The calculations using the $\delta = 0$ and $\delta = -1$ transformation could predict the presence of the Karman vortex behind the obstacle. However, the calculation using the $\delta = 1$ transformation could not. The main reason is that the coordinate lines are far from the boundary. From the above results, when the number of coordinate lines is limited for some practical reason such as the computing time and memory, the transformation based on a cylindrical Laplace equation is less preferred than the other two transformations.

IV. Conclusion

For two-dimensional axisymmetric flow problems, a cylindrical transformation based on an elliptic equation of a form like the stream function equation is more suitable than that based on a cylindrical Laplace equation.

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